

Generating Functional for Gauge Invariant Actions: Examples of Nonrelativistic Gauge Theories

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Abstract

We propose a generating functional for nonrelativistic gauge invariant actions. In particular, we consider actions without the usual magnetic term. Like in the Born-Infeld theory, there is an upper bound to the electric field strength in these gauge theories.

1 Introduction

The Wilson loops [1], with \mathcal{C} a closed curve, A the gauge field, and the trace taken in the N -dimensional fundamental representation of $SU(N)$

$$W(A, \mathcal{C}) = \frac{1}{N} \text{tr} P \exp \left[i \oint_{\mathcal{C}} dx^\mu A_\mu \right], \quad (1.1)$$

have been studied from many points of view in gauge theories [2].

The first thought about loops is the Wilson criterion for confinement. It is expected to show up in an area law for the expectation value of a loop

$$\mathcal{W}(\mathcal{C}) = \int [dA] e^{-S_{\text{YM}}} W(A, \mathcal{C}), \quad (1.2)$$

with S_{YM} the Yang-Mills action. The area law means that $\mathcal{W}(\mathcal{C})$ decays exponentially with an area enclosed by the loop. There have also been many discussions of this issue in the context of supersymmetric gauge theories with additional fermionic and scalar fields [2].

The second thought about loops is that instead of quantizing the gauge field A_μ , one can quantize coordinates x^μ . In terms of Euclidean path integral, that means

$$Z(A, \mathcal{C}) = \int [dx] e^{-S(x)} W(A, \mathcal{C}), \quad (1.3)$$

where $S(x)$ is an action for the quantized coordinates, fields x^μ with $\mu = 0, \dots, p$. For a $U(1)$ gauge group, such an integral appeared in the Feynman's first-quantized formulation of scalar

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QED many years ago [3]. Later, it also appeared in string theory, where the non-Abelian Wilson factor (1.1) was inserted into the Polyakov path integral to describe an effective action for massless string modes [4]. The main point here is that the renormalized path integral (1.3) is to be identified with the effective action¹

$$S(A) = Z(A). \quad (1.4)$$

Subsequent work [5, 7] has made it clear that this approach should be taken seriously, particularly in the context of D -brane actions [8].

In fact, one can consider the formula (1.3) as a generating functional for gauge invariant actions without any reference to string theory. In other words, given the action $S(x)$, one can use it to compute (generate) the corresponding gauge theory action via (1.3). Note that in general such actions include infinitely many terms. The two classical examples are that of Schwinger [9] and non-linear electrodynamics of Born-Infeld type obtained from (1.3) in [5].²

In this paper we have two main aims. First, we want to generalize the method to nonrelativistic gauge theories. In recent years, such theories have been of considerable interest in the context of strongly correlated electron systems as an effective field theory description of the long-wavelength physics at quantum critical points with anisotropic scaling. A peculiar property of those is that in the Lagrangian the usual \mathbf{H}^2 -term [10], or alternatively the \mathbf{E}^2 -term [11], has vanishing coefficient. For example, in $3 + 1$ dimensions the action is of the form [10]

$$S(\mathbf{E}, \mathbf{H}) = \int d^3x dt [\mathbf{E}^2 - \rho_4 (\nabla \times \mathbf{H})^2], \quad (1.5)$$

where $\mathbf{E}^2 = \mathbf{E} \cdot \mathbf{E}$ and ρ_4 is a coupling constant parameterizing a line of fixed points.

Our second aim is to find some (off-critical) non-linear deformations of the effective actions proposed in [10]. We keep in mind the Born-Infeld electrodynamics [12] that differs from that of Maxwell by higher order terms in the field strength. Moreover, as in the case of the Born-Infeld action where it describes the low energy dynamics of D -branes, such deformations might be useful for studying nonrelativistic branes.

Before getting to the specific examples that we will consider, let us set the basic framework. For the contour \mathcal{C} , we take a unit circle parameterized by an angular variable φ . Then, following [4], we split the field $x^\mu(\varphi)$ into a zero Fourier mode x_0^μ and remaining non-zero modes such that $x^\mu(\varphi) = x_0^\mu + \xi^\mu(\varphi)$ with $\int_0^{2\pi} \varphi \xi^\mu = 0$.³ A good pragmatic reason for doing so is that the x 's are interpreted as classical coordinates, while the ξ 's are quantum fluctuations which have to be integrated out. If we let $S(x)$ be quadratic in fluctuations, then, for a generic kinetic term, we can rewrite (1.3) as

$$S(A) = \int dx \int [d\xi] \exp \left[-\frac{1}{2} \xi G^{-1} \xi \right] W(A(x + \xi)), \quad (1.6)$$

where $\xi G^{-1} \xi = \int \int_0^{2\pi} d\varphi_1 d\varphi_2 \xi^\mu(\varphi_1) G_{\mu\nu}^{-1}(\varphi_1, \varphi_2) \xi^\nu(\varphi_2)$. We normalize the functional integral measure $[d\xi]$ as $\int [d\xi] \exp \left[-\frac{1}{2} \xi G^{-1} \xi \right] = 1$.

In conclusion, we make the following remarks about formula (1.6):

¹It may need some refinement in the presence of scalar fields. For a discussion see [6] and references therein.

²Some extrapolation formulas can be found in appendix of [7].

³In what follows we omit the index 0 from the zero mode when it is clear from the context.

(1) For $S(x) = \frac{1}{2} \int_0^{2\pi} d\varphi \dot{x} \dot{x}$ with $\dot{x} = \partial_\varphi x$ that describes the free propagation of a particle, G^{-1} is a local operator. Its Green function is given by $G(\varphi_{12}) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\varphi_{12}$, where $\varphi_{12} = \varphi_1 - \varphi_2$.

(2) For the "string" action $S(x) = \frac{1}{2} \int_D d^2\sigma \partial^a x^\mu \partial_a x_\mu$, with D a unit disc, G^{-1} is a non-local operator. Note that non-locality appears after Gaussian integration over $x(\sigma)$ in all internal points of the disc that provides an effective one-dimensional path integral [5]. In this case, the corresponding Green function is given by $G(\varphi_{12}) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \cos n\varphi_{12}$.

(3) Finally, there is one more issue to be mentioned here. The above construction seems somewhat formal that may be when the path integral diverges. Following [13], we will define one-dimensional path integral by using the Riemann ζ -function. Thus, we express all sums in terms of $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ and $\zeta(s, \frac{1}{2}) = \sum_{r=1/2}^{\infty} \frac{1}{r^s}$.

2 Gauge Invariant Action without Usual Magnetic Term

In this section we will describe a concrete example of nonrelativistic gauge theory without the usual magnetic F_{ij}^2 -term in the Lagrangian. Since the Wilson factor (1.1) is Lorentz invariant, we choose a non-invariant kinetic term

$$S(x) = \frac{1}{2} \int_0^{2\pi} d\varphi \left[\frac{1}{(2\pi\alpha')^2} \dot{x}^0 \dot{x}^0 + \frac{1}{2\pi\alpha'\kappa} (x^i - x_0^i)(x^i - x_0^i) \right], \quad (2.1)$$

or, equivalently,

$$\xi G^{-1} \xi = \frac{1}{(2\pi\alpha')^2} \xi^0 G_{00}^{-1} \xi^0 + \frac{1}{2\pi\alpha'\kappa} \xi^i G_{ij}^{-1} \xi^j. \quad (2.2)$$

Here

$$G^{00}(\varphi_{12}) = G_{12} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\varphi_{12}, \quad G^{ij}(\varphi_{12}) = \delta^{ij} G_{12} = \frac{1}{\pi} \delta^{ij} \sum_{n=1}^{\infty} \cos n\varphi_{12}, \quad (2.3)$$

where $\int_0^{2\pi} d\varphi_2 G_{12} G_{23}^{-1} = \delta^+(\varphi_{13})$. The function $\delta^+(\varphi)$ plays a role of the δ -function on the ξ 's. Therefore, it is given by $\delta^+(\varphi) = \frac{1}{\pi} \sum_{n=1}^{\infty} \cos n\varphi$. The indices i, j run from 1 to p . Like in string theory, we use $2\pi\alpha'$ for dimensional purposes. The normalization is designed to describe the gauge theory action with anisotropic scaling characterized by the dynamical critical exponent $z = 2$. In this case dimensions of all objects are measured in the units of spatial momenta such that $\dim x^0 = -2$ and $\dim x^i = -1$. κ is a relative factor of dimension zero. With such a choice, we define the zero mode measure as $dx = \tau_p dt d^p x$, where $t = x^0$, $\tau_p^{-1} = g(2\pi\alpha')^{(p+2)/2}$ and g is a dimensionless parameter.⁴ This provides $\dim dx = 0$.

Remark (1) above makes it clear that fluctuations in time direction are described by the usual particle term. The integral (1.6) for this case was intensively discussed in the literature.⁵ On the other hand, fluctuations in spacial directions are described by a sum of terms such that every term represents a white Gaussian noise.⁶ To our knowledge, there have been no studies of (1.6) with the noise term in the literature.

⁴In string theory g is proportional to the string coupling.

⁵For a review, see, e.g., [14] and references therein.

⁶Another possible way is to interpret them as a quadratic open string tachyon profile. However, a crucial difference with [15] is that the standard $\partial X^i \bar{\partial} X_i$ -term is missing in our case.

2.1 Leading Terms

To actually compute leading terms in the α' -expansion of (1.6), one would have to expand $A(x+\xi)$ in powers of ξ and then compute a few Feynman diagrams shown in Figure 1.⁷

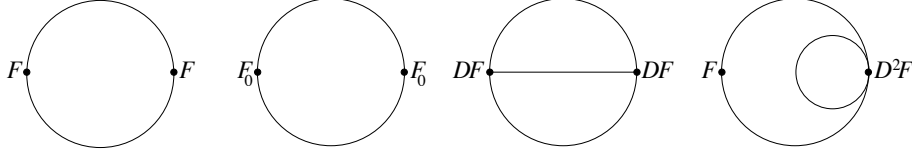


Figure 1: Feynman diagrams that contribute the leading terms in the α' -expansion. F_0 , F , D stand for F_{0i} , F_{ij} , D_i , respectively.

With $\zeta(-2) = 0$, the result, up to third order in α' , is given by

$$S(A) = \tau_p \int dt d^p x \operatorname{tr} \left[1 + (2\pi\alpha')^3 \left(b_1 F_{0i}^2 + b_2 (D_i F_{ij})^2 + b_3 F_{ij} F_{jk} F_{ki} \right) + O(\alpha'^4) \right], \quad (2.4)$$

$$b_1 = -\kappa \zeta(0), \quad b_2 = -\frac{1}{2\pi} \kappa^3 \zeta^2(-1), \quad b_3 = -\frac{3}{2} i \kappa^3 \zeta^2(-1).$$

Here $D_\mu = \partial_\mu - iA_\mu$ is the covariant derivative and $F_{\mu\nu} = i[D_\mu, D_\nu]$ is the field strength. With $\zeta(-1) = -1/12$ and $\zeta(0) = -1/2$, the above equation becomes

$$S(A) = \tau_p \int dt d^p x \operatorname{tr} \left[1 + 4\kappa(\pi\alpha')^3 \left(F_{0i}^2 - \frac{1}{144\pi} \kappa^2 (D_i F_{ij})^2 - \frac{1}{48} i \kappa^2 F_{ij} F_{jk} F_{ki} \right) + O(\alpha'^4) \right]. \quad (2.5)$$

In the Abelian case the F^3 term vanishes and therefore (2.5) reduces to

$$S(A) = \tau_p \int dt d^p x \left[1 + 4\kappa(\pi\alpha')^3 \left(F_{0i}^2 - \frac{1}{144\pi} \kappa^2 (\partial_i F_{ij})^2 \right) + O(\alpha'^4) \right]. \quad (2.6)$$

An important remark about (2.5) and (2.6) is that the α' -expansion doesn't coincide with a derivative expansion.

Motivated by the zero slope limit ($\alpha' \rightarrow 0$) of string theory, we would like to consider it in the problem at hand. As known, the role of this limit is just to remove the higher order corrections in α' from the string effective action and, as a result, it becomes quadratic in F .

So we take the limit $\alpha' \sim \varepsilon \rightarrow 0$, $g \sim \varepsilon^{2-(p/2)}$ with κ held fixed. Ignoring the constant term we get

$$S(A) = \rho \int dt d^p x \operatorname{tr} \left[F_{0i}^2 - \frac{1}{144\pi} \kappa^2 (D_i F_{ij})^2 - \frac{1}{48} i \kappa^2 F_{ij} F_{jk} F_{ki} \right], \quad (2.7)$$

⁷The diagrams look like those of [16] because in both cases the vertices are the same.

where $\rho = \frac{\kappa}{2g}(2\pi\alpha')^{2-(p/2)}$. In contrast to string theory, the resulting action now includes the $DFDF$ and F^3 -terms. It becomes purely quadratic in the field strength only in the Abelian case when the cubic term identically vanishes.

Finally, it remains to be seen how the action (1.5) is recovered. For this, we take $p = 3$ and use a description in terms of ordinary (Abelian) electric and magnetic fields: $\mathbf{E}_i = F_{0i}$ and $\mathbf{H}_k = \frac{1}{2}\epsilon_{kij}F_{ij}$. Then a little algebra shows that (2.6) does reduce to (1.5) with $\rho_4 = -\frac{\kappa^2}{144\pi}$.⁸

2.2 Slowly Varying Fields

We now turn to the question of what deformations of the action (1.5) are natural in our approach. Since (1.5) describes the long wavelength physics, it is natural to consider the case of slowly varying, but not necessarily small, fields. For this we will ignore higher derivative terms in the ξ expansion of $A(x + \xi)$.

For slowly varying Abelian fields, (1.6) reduces to

$$S(A) = \int dx \int [d\xi] \exp \left[-\frac{1}{2}\xi G^{-1}\xi + i \int_0^{2\pi} d\varphi \dot{\xi}^j \left(F_{0j}\xi^0 + \frac{1}{2}F_{ij}\xi^i + \frac{1}{3}\partial_k F_{ij}\xi^i\xi^k + \frac{1}{8}\partial_l\partial_k F_{ij}\xi^i\xi^k\xi^l \right) \right]. \quad (2.8)$$

It is convenient to integrate over ξ^0 first. Then, we get

$$S(A) = \int dx \int [d\xi^i] \exp \left[-\frac{1}{2}\xi^i \left(\frac{1}{2\pi\alpha'\kappa}\delta_{ij}\mathbf{G}^{-1} + (2\pi\alpha')^2 F_{0i}F_{0j}\ddot{\mathbf{G}} \right) \xi^j + i \int_0^{2\pi} d\varphi \xi^i \dot{\xi}^j \left(\frac{1}{2}F_{ij} + \frac{1}{3}\partial_k F_{ij}\xi^k + \frac{1}{8}\partial_l\partial_k F_{ij}\xi^k\xi^l \right) \right], \quad (2.9)$$

where $\ddot{\mathbf{G}} = \partial_1\partial_2\mathbf{G}(\varphi_{12})$.

As in string theory [17], it is possible to include the $F^2\xi\xi$ and $F\dot{\xi}\xi$ -terms into the propagator. However, for what follows we will include only one of them. To this end, we use the identity

$$\partial_1\partial_2\mathbf{G}(\varphi_{12}) = \mathbf{G}^{-1}(\varphi_{12}) \quad (2.10)$$

to rewrite (2.9) as

$$S(A) = \int dx \int [d\xi^i] \exp \left[-\frac{1}{4\pi\alpha'\kappa}\xi^i\mathcal{G}_{ij}\mathbf{G}^{-1}\xi^j + i \int_0^{2\pi} d\varphi \xi^i \dot{\xi}^j \left(\frac{1}{2}F_{ij} + \frac{1}{3}\partial_k F_{ij}\xi^k + \frac{1}{8}\partial_l\partial_k F_{ij}\xi^k\xi^l \right) \right], \quad (2.11)$$

where a new metric is given by

$$\mathcal{G}_{ij} = \delta_{ij} + (2\pi\alpha')^3\kappa F_{0i}F_{0j}. \quad (2.12)$$

Note that \mathcal{G} depends only on the electric field.

We are now ready to compute a few terms in $S(A)$ by treating all the magnetic terms under the integral in (2.11) as perturbations. With the metric \mathcal{G} held fixed, this is equivalent to a perturbation theory in α' . Up to third order, the result is

⁸In the process, one has to rescale the electric field $\mathbf{E} \rightarrow i\mathbf{E}$ by the Wick rotation.

$$S(A) = \tau_p \int dt d^p x \sqrt{\det(\delta_{ij} + (2\pi\alpha')^3 \kappa F_{0i} F_{0j})} \left[1 - \frac{1}{36\pi} (\pi\alpha'\kappa)^3 \partial_n F_{ij} \partial_m F_{kl} \mathcal{G}^{ni} \mathcal{G}^{mk} \mathcal{G}^{jl} + O(\alpha'^4) \right]. \quad (2.13)$$

Since we ignore derivatives of F_{0i} , we replaced the covariant derivatives (with respect to the metric \mathcal{G}_{ij}) with ordinary ones.

To compare (2.13) to (2.6), it is sufficient to expand $\sqrt{\det \mathcal{G}}$ and \mathcal{G}^{-1} in α' . Since we are only interested in the leading terms, we substitute $\sqrt{\det \mathcal{G}} = 1 + 4(\pi\alpha')^3 \kappa F_{0i}^2$ and $\mathcal{G}^{ij} = \delta^{ij}$ into (2.13) that immediately leads to the desired result.

Now, specializing to four dimension and ignoring the higher α' -corrections, we have

$$S(A) = \tau_3 \int dt d^3 x \sqrt{\det(\delta_{ij} + (2\pi\alpha')^3 \kappa \mathbf{E}_i \mathbf{E}_j)} \left[1 - \frac{1}{36\pi} (\pi\alpha'\kappa)^3 \partial_n \mathbf{F}_{ij} \partial_m \mathbf{F}_{kl} \mathcal{G}^{ni} \mathcal{G}^{mk} \mathcal{G}^{jl} \right], \quad (2.14)$$

with $\mathcal{G}_{ij} = \delta_{ij} + (2\pi\alpha')^3 \kappa \mathbf{E}_i \mathbf{E}_j$ and $\mathbf{F}_{ij} = \epsilon_{ijk} \mathbf{H}_k$. It can be interpreted as a non-linear deformation of the original action (1.5) by a slowly varying electric field.

To illustrate the deformation (2.14), we extend the leading order expression in the limit $\alpha' \rightarrow 0$ (again with $\mathbf{E} \rightarrow i\mathbf{E}/\sqrt{\rho}$ and $\mathbf{H} \rightarrow \mathbf{H}/\sqrt{\rho}$) to next order, and find

$$\mathcal{S} = \int d^3 x dt \left[\mathbf{E}^2 - \rho_4 (\nabla \times \mathbf{H})^2 + \rho_\alpha \left(\frac{1}{2} (\mathbf{E}^2)^2 + \rho_4 (\nabla \times \mathbf{H})^2 \mathbf{E}^2 - 2\rho_4 (\mathbf{E} \cdot \nabla \times \mathbf{H})^2 - 4\rho_4 (\nabla \times \mathbf{H}) \cdot (\mathbf{E} \times (\mathbf{E} \cdot \nabla) \mathbf{H}) \right) \right], \quad (2.15)$$

where $\rho_\alpha = g(2\pi\alpha')^{\frac{5}{2}}$.

Finally, let us consider higher order terms in F_{ij} . We have already found that there is no F_{ij}^2 in the action. To extend this calculation to order F^{2n} , we must compute a connected Feynman diagram shown in Fig.2.⁹

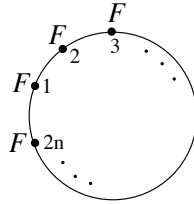


Figure 2: A Feynman diagram that contributes the F^{2n} -term to the action. F stands for F_{ij} .

It is easy to see that this diagram is proportional to the following integral

⁹Note that in the Abelian case there are no terms which are odd in F .

$$I_{2n} = \int_0^{2\pi} d\varphi_1 \dots \int_0^{2\pi} d\varphi_{2n} \dot{\mathbf{G}}_{12} \dots \dot{\mathbf{G}}_{2n1}, \quad (2.16)$$

where $\dot{\mathbf{G}}_{km} = \partial_k \mathbf{G}_{nm}$. Integrating over the φ 's, we get

$$I_{2n} = 2(-)^n \zeta(-2n) = 0. \quad (2.17)$$

We have used that $\zeta(-2n) = 0$ for any positive integer n . So, there are no $F\mathcal{G}^{-1}F \dots F\mathcal{G}^{-1}$ -terms in the action and, as a consequence, there are no $(\mathbf{H}^2)^n$ -terms for $p = 3$. This is the reason for not including the $F\dot{\xi}\xi$ -term into the propagator in (2.11).

2.3 Some Generalizations

We conclude this section with a computation that includes a more general form of the kinetic term. So we consider

$$\mathbf{G}_{12} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^a} \cos n\varphi_{12}, \quad \mathbf{G}_{12} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^b} \cos n\varphi_{12}, \quad (2.18)$$

where a and b are free parameters. Note that for $a = 2$ and $b = 0$, (2.18) reduces to (2.3).

To compute the F_{ij}^2 -term in the action, one has to evaluate the Feynman diagram as that of Fig. 2 with $n = 1$ and the propagator defined by (2.18). The integral (2.17) becomes in this case

$$I_2(b) = -2\zeta(2b - 2). \quad (2.19)$$

No essentially new computation is required to get the F_{i0}^2 -term. Replacing $F_{ij}\xi^i\dot{\xi}^j$ by $F_{0j}\xi^0\dot{\xi}^j$ leads to the following integral

$$\mathcal{I}_2(a, b) = \int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_2 \dot{\mathbf{G}}_{12} \dot{\mathbf{G}}_{21}, \quad (2.20)$$

which is simply

$$\mathcal{I}_2(a, b) = -2\zeta(a + b - 2). \quad (2.21)$$

Now we are ready to see what vanishing the F_{ij}^2 -term or alternatively the F_{0i}^2 -term in the action means. From (2.19) it follows that there is no F_{ij}^2 if $2b - 2$ coincides with one of the zeros of the Riemann zeta-function. For the trivial zeros, this restricts b to be a non-positive integer. For the non-trivial zeros, b is complex. It is given by $\frac{5}{4} + i\frac{t}{2}$.¹⁰ Similarly, the coefficient in front of the F_{0i}^2 -term vanishes if $a + b - 2$ is a zero of the zeta-function. This gives $a + b$ to be $2(1 - k)$ with k a positive integer for the trivial zeros and $\frac{5}{2} + it$ for the non-trivial zeros.

¹⁰According to the Riemann hypothesis, all the non-trivial zeros lie on the critical line $\frac{1}{2} + it$.

3 Examples with Fermions

The purpose of the present section is to add worldline fermions to the theory and do some explicit calculations illustrating our approach.

With the fermions, the Wilson factor (1.1) is typically extended to

$$\hat{W}(A, \mathcal{C}) = \frac{1}{N} \text{tr} P \exp \left[i \int_0^{2\pi} d\varphi \left(\dot{x}^\mu A_\mu - \frac{1}{2} F_{\mu\nu} \psi^\mu \psi^\nu \right) \right], \quad (3.1)$$

such that it respects the supersymmetry transformations

$$\delta x^\mu = \psi^\mu \epsilon, \quad \delta \psi^\mu = \dot{x}^\mu \epsilon, \quad (3.2)$$

even with a non-constant parameter ϵ .

If we formulate the theory in the superfield notation, the Wilson factor (3.1) can be written in the following form [18]

$$\hat{W}(A, \mathcal{C}) = \frac{1}{N} \text{tr} \left(\sum_{n=0}^{\infty} i^n \prod_{i=1}^n \int d\hat{\varphi}_i \mathcal{D} \hat{x}^\mu A_\mu(\hat{x}) \prod_{j=1}^{n-1} \Theta(\hat{\varphi}_{j,j+1}) \right). \quad (3.3)$$

Here

$$\hat{x}^\mu = x^\mu + \theta \psi^\mu, \quad \mathcal{D} = \theta \partial_\varphi - \partial_\theta, \quad d\hat{\varphi} = d\varphi d\theta, \quad \Theta(\hat{\varphi}_{ij}) = \Theta(\varphi_{ij}) + \theta_i \theta_j \delta(\varphi_{ij}), \quad (3.4)$$

where Θ is a step function.

Given the supersymmetry transformations, one can extend the action (2.1) to

$$\hat{S}(x, \psi) = \frac{1}{2} \int_0^{2\pi} d\varphi \left[\frac{1}{(2\pi\alpha')^2} (\dot{x}^0 \dot{x}^0 + \psi^0 \dot{\psi}^0) + \frac{1}{2\pi\alpha'\kappa} (x^i \dot{x}^i - \psi^i \partial^{-1} \dot{\psi}^i) \right] \quad (3.5)$$

by adding the fermions. Formally, it is invariant under (3.2). We consider for a moment the field x^i rather than its projection on non-zero modes. The fact that $\psi \partial^{-1} \dot{\psi}$ is a non-local operator is not in contradiction with our approach. The only restriction on $S(x, \psi)$ is that it must be quadratic in fluctuations.

Now we impose the antiperiodic boundary conditions on the fermions: $\psi^\mu(\varphi + 2\pi) = -\psi^\mu(\varphi)$. This implies that there are no fermionic zero modes and, therefore, the ψ 's can be interpreted as additional quantum fluctuations. Then it is natural to combine bosonic and fermionic fluctuations into a single superfield $\hat{\xi}^\mu = \xi^\mu + \theta \psi^\mu$ such that $\hat{x}^\mu = x_0^\mu + \hat{\xi}^\mu$.

With this choice of the boundary conditions we can write (3.5), with x_0^i subtracted from x^i , as

$$\xi G^{-1} \xi + \psi K^{-1} \psi = \frac{1}{(2\pi\alpha')^2} (\xi^0 G_{00}^{-1} \xi^0 + \psi^0 K_{00}^{-1} \psi^0) + \frac{1}{2\pi\alpha'\kappa} (\xi^i G_{ij}^{-1} \xi^j + \psi^i K_{ij}^{-1} \psi^j). \quad (3.6)$$

Here

$$K^{00}(\varphi_{12}) = K_{12} = \frac{1}{\pi} \sum_{r=1/2}^{\infty} \frac{1}{r} \sin r \varphi_{12}, \quad K^{ij}(\varphi_{12}) = \delta^{ij} K_{12} = \frac{1}{\pi} \delta^{ij} \sum_{r=1/2}^{\infty} r \sin r \varphi_{12}. \quad (3.7)$$

where $\int_0^\pi d\varphi_2 K_{12} K_{23}^{-1} = \delta^-(\varphi_{13})$. The function $\delta^-(\varphi)$ plays a role of the δ -function on the ψ 's. Explicitly, it is given by $\delta^-(\varphi_{12}) = \frac{1}{\pi} \sum_{r=1/2}^\infty \cos r\varphi_{12}$.

In terms of $\hat{\xi}$, (3.6) reads

$$\hat{\xi} \hat{G}^{-1} \hat{\xi} = \frac{1}{(2\pi\alpha')^2} \hat{\xi}^0 \hat{G}_{00}^{-1} \hat{\xi}^0 + \frac{1}{2\pi\alpha'\kappa} \hat{\xi}^i \hat{G}_{ij}^{-1} \hat{\xi}^j. \quad (3.8)$$

Here

$$\hat{G}^{00}(\varphi_{12}) = \hat{G}_{12} = G_{12} - \theta_1 \theta_2 K_{12}, \quad \hat{G}^{ij}(\varphi_{12}) = \delta^{ij} \hat{\mathbf{G}}_{12} = \delta^{ij} (\mathbf{G}_{12} - \theta_1 \theta_2 \mathbf{K}_{12}), \quad (3.9)$$

where $\hat{\xi} \hat{G}^{-1} \hat{\xi} = \iint d\hat{\varphi}_1 d\hat{\varphi}_2 \hat{\xi}^\mu(\varphi_1) \hat{G}_{\mu\nu}^{-1}(\varphi_1, \varphi_2) \hat{\xi}^\nu(\varphi_2)$ and $\int d\hat{\varphi}_2 \hat{G}_{12} \hat{G}_{23}^{-1} = \theta_1 \delta^-(\varphi_{13}) - \theta_3 \delta^+(\varphi_{13})$.¹¹

Once \hat{S} is found, we will use it in the next subsection to study the generating functional

$$S(A) = \int dx \int [d\hat{\xi}] \exp \left[-\frac{1}{2} \hat{\xi} \hat{G}^{-1} \hat{\xi} \right] \hat{W}(A(x + \hat{\xi})). \quad (3.10)$$

Here we normalize the functional integral measure $[d\hat{\xi}]$ as $\int [d\hat{\xi}] \exp \left[-\frac{1}{2} \hat{\xi} \hat{G}^{-1} \hat{\xi} \right] = 1$.

3.1 Leading Terms

We will now carry out a precisely analogous computation as that of subsection 2.1. The simplest way to do so is to use the superfield formalism. In practice, this means to replace $d\varphi$, $\Theta(\varphi)$, ∂_φ , G , etc. with $d\hat{\varphi}$, $\Theta(\hat{\varphi})$, \mathcal{D} , \hat{G} , etc. in the corresponding expressions for the Feynman diagrams shown in Figure 1. Using $\zeta(-2) = \zeta(-2, \frac{1}{2}) = 0$, the diagrams can be evaluated to give

$$S(A) = \tau_p \int dt d^p x \operatorname{tr} \left[1 + (2\pi\alpha')^3 \left(s_1 F_{0i}^2 + s_2 (D_i F_{ij})^2 + s_3 F_{ij} F_{jk} F_{ki} \right) + O(\alpha'^4) \right],$$

$$s_1 = -\kappa \left(\zeta(0) - \zeta(0, \frac{1}{2}) \right), \quad s_2 = -\frac{1}{2\pi} \kappa^3 \left(\zeta(-1) - \zeta(-1, \frac{1}{2}) \right)^2, \quad s_3 = -\frac{3}{2} i \kappa^3 \left(\zeta(-1) - \zeta(-1, \frac{1}{2}) \right)^2. \quad (3.11)$$

With $\zeta(0, \frac{1}{2}) = 0$ and $\zeta(-1, \frac{1}{2}) = 1/24$, (3.11) takes the form

$$S(A) = \tau_p \int dt d^p x \operatorname{tr} \left[1 + 4\kappa(\pi\alpha')^3 \left(F_{0i}^2 - \frac{1}{64\pi} \kappa^2 (D_i F_{ij})^2 - \frac{3}{64} i \kappa^2 F_{ij} F_{jk} F_{ki} \right) + O(\alpha'^4) \right]. \quad (3.12)$$

A few noteworthy facts are the following. The effect of the fermions on the coefficients b_i is a shift: $\zeta(s) \rightarrow \zeta(s) - \zeta(s, \frac{1}{2})$. The only exception is the coefficient in front of the quadratic term. In fact, it doesn't get shifted because $\zeta(s, \frac{1}{2}) = 0$ for $s = 0$. In the Abelian case the F^3 term vanishes. After taking the limit $\alpha' \rightarrow 0$ and setting $p = 3$, the action reduces to (1.5) with $\rho_4 = -\frac{\kappa^2}{64\pi}$.

¹¹Note that $\hat{G}_{12}^{-1} = K_{12}^{-1} - \theta_1 \theta_2 G_{12}^{-1}$.

3.2 Slowly Varying Fields

Now let us explore some issues that arise in the case of slowly varying Abelian fields. We use again the superfield formalism.

For this case, (3.10) reduces to

$$S(A) = \int dx \int [d\hat{\xi}] \exp \left[-\frac{1}{2} \hat{\xi} \hat{G}^{-1} \hat{\xi} + i \int d\hat{\varphi} \mathcal{D} \hat{\xi}^j \left(F_{0j} \hat{\xi}^0 + \frac{1}{2} F_{ij} \hat{\xi}^i + \frac{1}{3} \partial_k F_{ij} \hat{\xi}^i \hat{\xi}^k + \frac{1}{8} \partial_l \partial_k F_{ij} \hat{\xi}^i \hat{\xi}^k \hat{\xi}^l \right) \right]. \quad (3.13)$$

Integrating out $\hat{\xi}^0$, one gets simply

$$S(A) = \int dx \int [d\hat{\xi}^i] \exp \left[-\frac{1}{2} \hat{\xi}^i \left(\frac{1}{2\pi\alpha'\kappa} \delta_{ij} \hat{\mathbf{G}}^{-1} - (2\pi\alpha')^2 F_{0i} F_{0j} \mathcal{D}_1 \mathcal{D}_2 \hat{\mathbf{G}} \right) \hat{\xi}^j \right. \\ \left. + i \int d\hat{\varphi} \hat{\xi}^i \mathcal{D} \hat{\xi}^j \left(\frac{1}{2} F_{ij} + \frac{1}{3} \partial_k F_{ij} \hat{\xi}^k + \frac{1}{8} \partial_l \partial_k F_{ij} \hat{\xi}^k \hat{\xi}^l \right) \right], \quad (3.14)$$

where $\hat{\xi} \mathcal{D}_1 \mathcal{D}_2 \hat{\mathbf{G}} \hat{\xi} = \int \int d\hat{\varphi}_1 d\hat{\varphi}_2 \hat{\xi}_1 \mathcal{D}_1 \mathcal{D}_2 \hat{\mathbf{G}}_{12} \hat{\xi}_2$.

It is easy to see now that

$$\mathcal{D}_1 \mathcal{D}_2 \hat{\mathbf{G}}(\varphi_{12}) = -\hat{\mathbf{G}}^{-1}(\varphi_{12}) \quad (3.15)$$

is a superspace version of (2.10). The minus sign is due to the fact that the \mathcal{D} 's anticommute with each other. This identity allows us to combine the two first terms of (3.14) into a single one. So, we have

$$S(A) = \int dx \int [d\hat{\xi}^i] \exp \left[-\frac{1}{4\pi\alpha'\kappa} \hat{\xi}^i \mathcal{G}_{ij} \hat{\mathbf{G}}^{-1} \hat{\xi}^j + i \int d\hat{\varphi} \hat{\xi}^i \mathcal{D} \hat{\xi}^j \left(\frac{1}{2} F_{ij} + \frac{1}{3} \partial_k F_{ij} \hat{\xi}^k + \frac{1}{8} \partial_l \partial_k F_{ij} \hat{\xi}^k \hat{\xi}^l \right) \right], \quad (3.16)$$

with \mathcal{G}_{ij} defined in (2.12).

Having derived (3.16), we are now in a position to compute a few terms in the action. Just as in section 2.2, keeping the metric \mathcal{G} fixed, we find up to third order in α'

$$S(A) = \tau_p \int dt d^p x \sqrt{\det(\delta_{ij} + (2\pi\alpha')^3 \kappa F_{0i} F_{0j})} \left[1 - \frac{1}{16\pi} (\pi\alpha'\kappa)^3 \partial_n F_{ij} \partial_m F_{kl} \mathcal{G}^{ni} \mathcal{G}^{mk} \mathcal{G}^{jl} + O(\alpha'^4) \right]. \quad (3.17)$$

To get from (3.17) to (3.12), its Abelian version, we must expand \mathcal{G} in α' . In doing so, the F_{0i}^2 term comes from the determinant of \mathcal{G} , while the $(DF)^2$ term is already in (3.17).

We conclude this section by computing the higher order terms in F_{ij} . For this, we need to evaluate the Feynman diagram of Figure 2 in the superfield formalism. This diagram is given, up to a constant multiple, by a multiple integral

$$\hat{I}_{2n} = \int d\hat{\varphi}_1 \dots \int d\hat{\varphi}_{2n} \mathcal{D}_1 \hat{\mathbf{G}}_{12} \dots \mathcal{D}_{2n} \hat{\mathbf{G}}_{2n1}. \quad (3.18)$$

Performing integration, we obtain

$$I_{2n} = 2(-)^{n+1}(\zeta(-2n) - \zeta(-2n, \frac{1}{2})) = 0. \quad (3.19)$$

We have used that $\zeta(-2n) = \zeta(-2n, \frac{1}{2}) = 0$ for any positive integer n . Thus, there are no $F\mathcal{G}^{-1}F \dots F\mathcal{G}^{-1}$ terms in the action and, as a consequence, there are no $(\mathbf{H}^2)^n$ -terms for $p = 3$.

3.3 Some Further Generalizations

To generalize the discussion of section 2.3 to the case with the fermions, the first step is to pick up fermionic kinetic terms such that

$$\mathbf{K}_{12} = \frac{1}{\pi} \sum_{r=1/2}^{\infty} \frac{1}{r^{a-1}} \sin r\varphi_{12}, \quad \mathbf{K}_{12} = \frac{1}{\pi} \sum_{r=1/2}^{\infty} \frac{1}{r^{b-1}} \sin r\varphi_{12}. \quad (3.20)$$

So for $a = 2$ and $b = 0$, (3.20) coincides with (3.7). Then, the corresponding superspace propagators are constructed by combining them with the G 's of section 2.3.

Now, to compute a coefficient in front of the F_{ij}^2 -term in the action, one has to evaluate the Feynman diagram of Figure 2 with $n = 1$. In the case of interest the integral (3.18) becomes

$$\hat{I}_2(b) = 2(\zeta(2b-2) - \zeta(2b-2, \frac{1}{2})) = 4(1 - 2^{2b-3})\zeta(2b-2). \quad (3.21)$$

Here we have used the identity that $\zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s)$.

Similarly, one can find a coefficient in front of the F_{0i}^2 -term. Replacing $F_{ij}\hat{\xi}^i\mathcal{D}\hat{\xi}^j$ by $F_{0j}\hat{\xi}^0\mathcal{D}\hat{\xi}^j$ leads to the following integral

$$\hat{\mathcal{I}}_2(a, b) = \int d\hat{\varphi}_1 \int d\hat{\varphi}_2 \mathcal{D}_1 \hat{\mathbf{G}}_{12} \mathcal{D}_2 \hat{\mathbf{G}}_{21}. \quad (3.22)$$

After performing integration, one gets

$$\hat{\mathcal{I}}_2(a, b) = 2(\zeta(a+b-2) - \zeta(a+b-2, \frac{1}{2})) = 4(1 - 2^{a+b-3})\zeta(a+b-2). \quad (3.23)$$

Clearly, no new analysis is required to conclude what vanishing the F_{ij}^2 and F_{0i}^2 -terms in the action means. Therefore our conclusions here are similar to those of section 2.3.

4 Concluding Comments

We will here conclude with brief observations about our approach and the non-relativistic gauge invariant actions derived above.

(1) For $F_{ij} = 0$, the Abelian actions (2.13) and (3.17) reduce to

$$S(A) = \tau_p \int dt d^p x \sqrt{\det(\delta_{ij} + F_{0i}F_{0j})}. \quad (4.1)$$

In order to keep the equations simple we have set $(2\pi\alpha')^3\kappa = 1$. Clearly, the α' -dependence can be easily restored on dimensional grounds. It is surprising that (4.1) can be written in the Born-Infeld form

$$S(A) = \tau_p \int dt d^p x \sqrt{\det(\delta_{\mu\nu} + F_{\mu\nu})}. \quad (4.2)$$

To show this, we derive a useful identity. We do that in a frame where $F_{\mu\nu}$ has a special form

$$F = \begin{pmatrix} 0 & F_{01} & 0 & \dots & 0 \\ -F_{01} & 0 & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix}, \quad (4.3)$$

with F_{01} the only nontrivial component. Then, we find

$$\det(\delta_{ij} + F_{0i}F_{0j}) = \det(\delta_{\mu\nu} + F_{\mu\nu}) = 1 + F_{01}^2. \quad (4.4)$$

The relation with the Born-Infeld theory implies that, after Wick rotating to Lorentzian signature, there is an upper bound to the electric field strength. If one restores dimensions, the critical field strength E_c is given by

$$E_c = \frac{1}{\sqrt{(2\pi\alpha')^3\kappa}}. \quad (4.5)$$

Note that, as in string theory, there is no upper bound in the limit $\alpha' \rightarrow 0$.

These similarities with the Born-Infeld action suggest a possibility of considering the actions we found above as candidates for effective actions of nonrelativistic branes.

(2) As noted above, in the presence of fermions $\zeta(s)$ combines with $\zeta(s, \frac{1}{2})$ to form a linear combination $\zeta(s) - \zeta(s, \frac{1}{2})$. It is very special in that it is a holomorphic function of the complex variable s . A simple pole of $\zeta(s)$ at $s = 1$ cancels with that of $\zeta(s, \frac{1}{2})$. Indeed, one can easily check that $\lim_{s \rightarrow 1} (\zeta(s) - \zeta(s, \frac{1}{2})) = -\ln 4$. This means that the Feynman diagrams we considered in section 3 remain finite for all values of a and b . As a result, the path integral representation for the action (3.10) is well-defined at least up to third order in α' .

(3) Although the Gaussian noise terms describing fluctuations in spatial directions were introduced by hand so as to recover finally the gauge theory action (1.5), there is some indirect evidence for this. A few facts are particularly interesting:

(i) This gauge theory action was proposed in [10] to describe the Rokhsar-Kivelson points of quantum dimer models.

(ii) Recently, it was claimed in [19] that the quantization scheme used for the quantum dimer models is nothing else but a discrete analog of stochastic quantization.

(iii) As known, the notion of noise plays a pivotal role in stochastic quantization.

If so, then it seems natural to expect the appearance of "noise" in our approach too. As we have shown above this is indeed the case.

(4) It is worth mentioning that a non-Abelian gauge theory action without the usual magnetic term was also proposed in [20].¹² It differs from ours (2.7) and (3.12) by the absence of the F^3 -term. From our approach we don't see any reason for dropping it. It disappears in the open superstring effective action ($a = b = 1$), where, on the other hand, the magnetic term is present because of Lorentz invariance.

¹²Note that a non-Abelian action without the usual electric term was proposed in [11].

(5) So far, stringy motivated methods were developed only for relativistic field theories [14]. In this paper we have taken a first step to develop them for nonrelativistic gauge theories by deriving some gauge invariant actions. Clearly, this avenue of research deserves to be pursued. In addition to string theory, it might be quite important for nonrelativistic QCD, statistical and solid state physics.

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